

An Efficient Method for Locating and Computing Periodic Orbits of Nonlinear Mappings*

MICHAEL N. VRAHATIS

Department of Mathematics, University of Patras, GR-261.10 Patras, Greece

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$$X = \Phi_2^p(X) \equiv \underbrace{\Phi_2(\Phi_2(\cdots (\Phi_2(X)) \cdots))}_{p \text{ times}}. \quad (1.2)$$

The accurate computation of periodic orbits of nonlinear mappings and the precise knowledge of their properties are very important for studying the behavior of many dynamical systems of physical interest. In this paper, we present an efficient numerical method for locating and computing to any desired accuracy periodic orbits (stable, unstable, and complex) of any period. The method described here is based on the topological degree of the mapping and is particularly useful, since the only computable information required is the algebraic signs of the components of the mapping. This method always converges rapidly to a periodic orbit independently of the initial guess and is particularly useful when the mapping has many periodic orbits, stable and unstable, close to each other, all of which are desired for the application. We illustrate this method first on a two-dimensional quadratic mapping, used in the study of beam dynamics in particle accelerators, to compute rapidly and accurately its periodic orbits of periods $p = 1, 5, 16, 144, 1296, 10368$ and then obtain periodic orbits of its four-dimensional complex version for periods which also reach up to the thousands. © 1995 Academic Press, Inc.

1. INTRODUCTION

It is well known that two-dimensional mappings of the (x_1, x_2) plane onto itself defined by

$$\Phi_2: \begin{cases} \hat{x}_1 = \varphi_1(x_1, x_2), \\ \hat{x}_2 = \varphi_2(x_1, x_2) \end{cases} \quad (1.1)$$

can be used for the study of many dynamical systems with two degrees of freedom. Such mappings can model conservative or dissipative dynamical systems, depending on whether the mapping is area-preserving or area-contracting, respectively [7, 8, 23, 26, 27]. For the analysis of these mappings a central role is played by points which are *invariant* or *fixed* under the mapping and are more commonly called *periodic orbits* of the mapping [45]. We say that $X = (x_1, x_2)^T$ is a fixed point of Φ_2 if $\Phi_2(X) = X$ and a fixed point of *order* p , or a periodic orbit of *period* p if

In general, analytic expressions for evaluating periodic orbits are available only if the mapping is a polynomial of low degree and the period is low (for example, see [27]). On the other hand, it is difficult to find in the literature efficient methods for computing orbits of high period if the mapping is not decomposable into involutions [17, 23, 38]. Also, traditional iterative schemes such as Newton's method and related classes of algorithms [10–12, 15, 16, 36, 37] often fail, since they converge to a fixed point almost independently of the initial guess, while there exist several fixed points close to each other, which are all desirable for the applications [2]. Moreover, these methods are affected by the mapping evaluations taking large values in the neighborhood of periodic orbits which are unstable in the linear approximation and are called *unstable* or *saddle-hyperbolic* periodic orbits [26]. Also, in general, these methods often fail due to the nonexistence of derivatives or poorly behaved partial derivatives near fixed points [10–12, 15, 16, 36, 37].

In this paper, we describe an efficient numerical method for computing rapidly periodic orbits (be they stable, unstable or complex) of any period and to any desired accuracy. This method exploits topological degree theory to provide a criterion for the existence of a periodic orbit of an iterate of the mapping within a given region. More specifically, the method constructs a polyhedron in such a way that the value of the topological degree of an iterate of the mapping relative to this polyhedron is ± 1 , which means that there exists a periodic orbit within this polyhedron. Then it repeatedly subdivides its sides (and its diagonals) so that the new polyhedron also retains this property (of the existence of a periodic orbit within its interior) without making any computation of the topological degree. These subdivisions take place iteratively until a periodic orbit is computed to a predetermined accuracy. After one periodic point of the orbit has been computed, the method iterates the mapping to obtain all the other points of the same period to the same accuracy. More specifically, the method checks whether each mapping iteration gives a periodic point (of the same period) to the predetermined accuracy. If so, the method

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continues with the next iteration; otherwise it applies the process of subdivisions to a smaller polyhedron which contains the approximate periodic point. The vertices of this small polyhedron can be easily selected by permuting the components of the approximate periodic point.

This method becomes especially significant for the computation of high period orbits (stable or unstable), where other more traditional approaches (like Newton's method, etc.) cannot easily distinguish among the closely neighboring periodic orbits. Moreover, this method is particularly useful, since the only computable information it requires is the algebraic signs of the components of the mapping. Thus it is not affected by the mapping evaluations taking large values in neighborhoods of unstable periodic orbits.

Here, we shall illustrate this method first on Hénon's quadratic area-preserving two-dimensional mapping [8, 19, 26, 27]

$$\Phi_2: \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = R(a) \begin{pmatrix} x_1 \\ x_2 + g(x_1) \end{pmatrix}, \quad (1.3)$$

where

$$(x_1, x_2)^T \in \mathbb{R}^2, \quad (1.4)$$

$$R(a) = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}, \quad a \in [0, \pi], \quad g(x_1) = -x_1^2,$$

to compute rapidly and accurately the periodic orbits for given values of a and various periods $p = 1, 5, 16, 144, 1296, 10368$. Next we shall compute complex periodic orbits, for periods which reach up to the thousands, of the Hénon mapping expanding (1.3) to four dimensions by separating its real and imaginary parts. Thus we shall study the four-dimensional mapping

$$\Phi_4: \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} = \begin{pmatrix} R(a) & \mathbb{O} \\ \mathbb{O} & R(a) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 - x_1^2 + y_1^2 \\ y_1 \\ y_2 - 2x_1y_1 \end{pmatrix}, \quad (1.5)$$

which is closely related to a quadratic mapping in four dimensions used in the simulation of beam dynamics in particle accelerators [4, 9].

In the next section we give a criterion for the existence of a periodic orbit within a scaled translation of the unit cube. Also in that section we describe in detail the idea behind this criterion and how it works.

Then, in Section 3, we describe a generalized bisection method used in combination with the above criterion to compute rapidly a periodic orbit to any predetermined accuracy and period.

The procedure for computing all the periodic points of orbits

of the same period according to the predetermined accuracy is presented in Section 4. Also, in that section, these procedures are applied to the calculation of stable and unstable periodic orbits of the two-dimensional Hénon's mapping for periods which reach up to the thousands.

Applications of these methods to periodic orbits of the four-dimensional mapping (1.5) are presented in Section 5. Also, in the same section, an application of these ideas to a four-dimensional symplectic form of Hénon's mapping is discussed. We finally end, in Section 6, with some concluding remarks and a short discussion of these methods.

2. A TOPOLOGICAL DEGREE CRITERION FOR THE EXISTENCE OF A PERIODIC ORBIT

In this section, we shall implement topological degree theory to give a criterion for the existence of a periodic orbit within a given region of the phase space of the system. This criterion is based on the construction of a "characteristic polyhedron" within a scaled translation of the unit cube. The concept of a characteristic polyhedron will be reviewed and a procedure for its construction will be presented. The theoretical development of the concepts employed here can be found in [46–48, 50, 55].

As we said previously, the problem of finding a periodic orbit of a nonlinear mapping $\Phi_n = (\varphi_1, \varphi_2, \dots, \varphi_n): \mathcal{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ from a domain \mathcal{D} into \mathbb{R}^n is to find a point $X^* = (x_1^*, x_2^*, \dots, x_n^*)^T \in \mathcal{D}$ which satisfies

$$\Phi_n(X^*) = X^*, \quad (2.1)$$

or, equivalently,

$$\begin{aligned} \varphi_1(x_1^*, x_2^*, \dots, x_n^*) &= x_1^*, \\ \varphi_2(x_1^*, x_2^*, \dots, x_n^*) &= x_2^*, \\ &\vdots \\ \varphi_n(x_1^*, x_2^*, \dots, x_n^*) &= x_n^*. \end{aligned} \quad (2.2)$$

Obviously, the problem of finding a periodic orbit is equivalent to the problem of solving

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= 0, \\ f_2(x_1, x_2, \dots, x_n) &= 0, \\ &\vdots \\ f_n(x_1, x_2, \dots, x_n) &= 0, \end{aligned} \quad (2.3)$$

by considering the mapping $F_n = (f_1, f_2, \dots, f_n) = \Phi_n - I_n$ (where I_n indicates the identity mapping) instead of Φ_n and solving

$$F_n(X) = \mathbb{O}_n \quad (2.4)$$

(where $\Theta_n = (0, 0, \dots, 0)^T$ indicates the origin of \mathbb{R}^n) instead of Eq. (2.1).

Many problems require the solution of systems of nonlinear equations for which Newton's method and related classes of algorithms [10–12, 15, 16, 36, 37] fail, due to nonexistence of derivatives or poorly behaved partial derivatives. Also, Newton's method, as well as Newton-like methods [10–12, 15, 16, 37], often converge to a solution X^* of $F_n(X) = \Theta_n$ almost independently of the initial guess, while there may exist several solutions, all of which are desired for the application [2]. Because of these reasons, various approaches based upon topological degree theory and, in particular, generalized bisection methods have been investigated in the past few years [18, 24, 28–30, 41, 46–50, 52, 55].

Bisection methods for finding solutions of systems of equations depend on a "criterion," which can guarantee that a solution will lie within a given region. Then this region can be subdivided in such a way that the criterion can again be applied within the new refined region.

In one dimension, this criterion consists of the product of the signs of the function evaluations at the endpoints of a given interval. Specifically, if we desire to locate a solution of an equation $f(x) = 0$ in the interval (a, b) , where $f: (a, b) \subset \mathbb{R} \rightarrow \mathbb{R}$ is continuous we can examine whether the relation

$$\operatorname{sgn} f(a) \cdot \operatorname{sgn} f(b) = -1 \quad (2.5)$$

is fulfilled, where sgn is the well-known sign function with the values

$$\operatorname{sgn} \psi = \begin{cases} -1, & \text{if } \psi < 0, \\ 0, & \text{if } \psi = 0, \\ 1, & \text{if } \psi > 0; \end{cases} \quad (2.6)$$

and if so, then we certainly know that there is at least one solution within (a, b) . This criterion is known as *Bolzano's existence criterion*.

Instead of Bolzano's criterion one may also use the following criterion:

$$\deg[f, (a, b), 0] = \frac{1}{2} \{ \operatorname{sgn} f(b) - \operatorname{sgn} f(a) \}. \quad (2.7)$$

The $\deg[f, (a, b), 0]$ is called the *topological degree of f at the origin relative to (a, b)* . Now if the value of $\deg[f, (a, b), 0]$ is not zero we know with certainty that there is at least one solution in (a, b) . Note that if the value of $\deg[f, (a, b), 0]$ is not zero then Bolzano's criterion is fulfilled. The value of $\deg[f, (a, b), 0]$ gives additional information concerning the behavior of the solutions of $f(x) = 0$ in (a, b) relative to the slopes of f [24]. For example, if $\deg[f, (a, b), 0] = 1$ which means that $f(b) > 0$ and $f(a) < 0$, then the number of solutions at points,

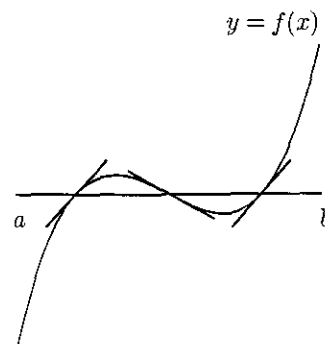


FIG. 1. Interval for which the Bolzano's criterion is fulfilled and $\deg[f, (a, b), 0] = 1$.

where $f(x)$ has a positive slope, exceeds by one the number of solutions at points at which $f(x)$ has a negative slope (see Fig. 1).

The criterion of topological degree, as well as Bolzano's criterion, transfers all the information regarding the roots to the boundary of the given region. Now, using the value of the topological degree, or Bolzano's criterion, we are able to calculate a solution of $f(x) = 0$ by bisecting the interval (a, b) . So we subdivide (a, b) into two intervals (a, c) , $[c, b)$, where $c = (a + b)/2$ is the midpoint of (a, b) , and we keep the subinterval for which the value of the topological degree is not zero, relative to itself, by checking the information on the boundaries. In this way we keep at least one solution within a smaller interval. We can continue this procedure to approximate a solution until the endpoints of the final subinterval differ from each other by less than a fixed amount. This method is called *bisection* or *Bolzano's method* and can be written as [20–22, 47, 48, 50, 52, 54–56]

$$\begin{aligned} x_{n+1} &= x_n + \operatorname{sgn} f(a) \cdot \operatorname{sgn} f(x_n) \cdot (b - a)/2^{n+1}, \\ x_0 &= a, \quad n = 0, 1, \dots \end{aligned} \quad (2.8)$$

Of course, it converges to a solution x^* in (a, b) if for some x_n , $n = 1, 2, \dots$, we have

$$\operatorname{sgn} f(x_0) \cdot \operatorname{sgn} f(x_n) = -1. \quad (2.9)$$

Also, the minimum number of iterations γ , which are required to obtain an approximate solution x' such that $|x' - x^*| \leq \varepsilon$ for some $\varepsilon \in (0, 1)$ is given by

$$\gamma = \lceil \log_2((b - a) \cdot \varepsilon^{-1}) \rceil, \quad (2.10)$$

where the notation $\lceil \cdot \rceil$ refers to the smallest integer not less than the real number quoted.

Based on the relation (2.10) it is proven in [40] that the bisection method is *optimal*, i.e., that it possesses asymptotically the best rate of convergence. Also, we would like to mention

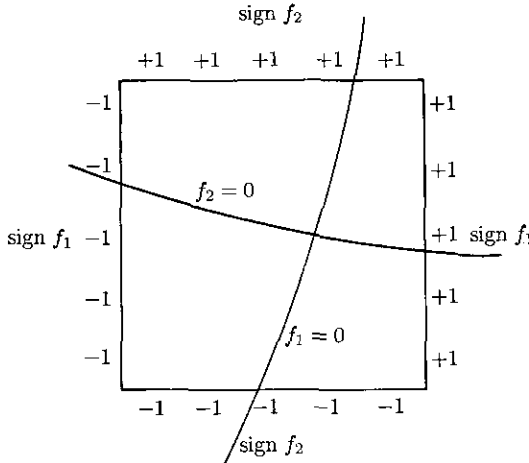


FIG. 2. Miranda's box.

here that the only computable information required by the bisection method is the signs of various function evaluations, and, as it always converges within the given interval (a, b) independently of its length, it is a global convergence method. Also, using the relation (2.10), we can easily find out the number of iterations needed for the attainment of an approximate solution to a predetermined accuracy beforehand.

It would be very desirable, of course, to achieve a generalization of the above bisection method to higher dimensions. To do this we have to generalize first Bolzano's or a similar topological degree criterion. A straightforward extension of Bolzano's criterion to two dimensions, to locate a root of a continuous mapping $F_2 = (f_1, f_2): \mathcal{D} \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$, is to consider a box such that the function $f_1(x_1, x_2)$ may possess opposite signs at the opposite sides and the function $f_2(x_1, x_2)$ may also possess opposite signs at the remaining sides of the box (see Fig. 2). Then, if this criterion is satisfied there is at least one solution of the system:

$$\begin{aligned} f_1(x_1, x_2) &= 0, \\ f_2(x_1, x_2) &= 0, \end{aligned} \tag{2.11}$$

within the given box. This box, which is known as Miranda's 2-cube [34, 41, 53], can be extended to higher dimensions to provide the criterion of existence. Note that Miranda's criterion transfers all the information regarding the roots to the boundary of the given region. However, although Miranda's box always gives a criterion for the existence of a root, its construction is often impossible and cannot be applied for the computation of a root. To overcome these difficulties, we give a "suitable" polyhedron whose construction guarantees the existence of at least one root in its interior. Before presenting the concept of this polyhedron, let us give a generalization of the topological

degree on which various generalized methods of bisections are based. According to these methods, one establishes the existence of at least one solution of a system $F_n(X) = \Theta_n$, where $F_n = (f_1, f_2, \dots, f_n): \mathcal{D} \rightarrow \mathbb{R}^n$, is continuous on the closure $\overline{\mathcal{D}}$ of \mathcal{D} such that $F_n(X) \neq \Theta_n$ for X on the boundary $\partial\mathcal{D}$ of \mathcal{D} by computing the topological degree of F_n at Θ_n relative to \mathcal{D} . This is denoted by $\text{deg}[F_n, \mathcal{D}, \Theta_n]$ and can be defined by the sum

$$\text{deg}[F_n, \mathcal{D}, \Theta_n] = \sum_{X \in F_n^{-1}(\Theta_n)} \text{sgn det } J_{F_n}(X), \tag{2.12}$$

where $\text{det } J_{F_n}$ indicates the determinant of the Jacobian matrix. Now, if a nonzero value of $\text{deg}[F_n, \mathcal{D}, \Theta_n]$ is obtained, then by Kronecker's existence theorem [36, p. 161] it follows that there is at least one solution of system $F_n(X) = \Theta_n$ within \mathcal{D} . On the other hand, if $\text{deg}[F_n, \mathcal{D}, \Theta_n] = 0$, no conclusions can be drawn because more information about F_n is needed [1, 13, 36, 44].

However, although the nonzero value of $\text{deg}[F_n, \mathcal{D}, \Theta_n]$ plays an important role in the existence of a solution, its exact value is useless, since it does not give any additional information about the existence of the solutions of system $F_n(X) = \Theta_n$. Moreover, the computation of $\text{deg}[F_n, \mathcal{D}, \Theta_n]$ is a time-consuming procedure [28, 30] and cannot be accurately achieved unless the modulus of continuity of F_n on \mathcal{D} is known [42, 44]. For more details about degree theory we refer the reader to [1, 5, 6, 13, 14, 25, 28, 30, 32, 35, 36, 39, 41–44]. The bisection method, on the other hand, which is described below, implements the concept of a characteristic n -polyhedron and avoids all calculations concerning the topological degree. This is because, under suitable assumptions on its boundary, a characteristic n -polyhedron, always contains at least one solution of system $F_n(X) = \Theta_n$, since the absolute value of the topological degree of F_n at Θ_n relative to it is always equal to one [46, 50, 55].

In what follows, we give the concept of a characteristic n -polyhedron as well as an algorithm for its construction. To define it we have to give some other tools. Consider first the n -digit binary representation $b_{n-1}^i b_{n-2}^i \dots b_1^i b_0^i, b_j^i \in \{0, 1\}$ of an integer number $i \in \{0, 1, \dots, 2^n - 1\}$ such that

$$i = b_{n-1}^i 2^{n-1} + b_{n-2}^i 2^{n-2} + \dots + b_1^i 2^1 + b_0^i 2^0. \tag{2.13}$$

Consider also the n -complete matrices defined by

$$\begin{aligned} \mathcal{M}_n &= [C_{i,j}], \quad C_{i,j} = 2b_{n-j}^{i-1} - 1, \quad i = 1, 2, \dots, 2^n, \\ & \quad j = 1, 2, \dots, n. \end{aligned} \tag{2.14}$$

For example for $n = 1, 2, 3$ we have respectively

$$\mathcal{M}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathcal{M}_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix},$$

$$\mathcal{M}_3 = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}. \quad (2.15)$$

Now, if $n = 1$ we consider the segment $[x_1, x_2]$ and evaluate the sign of $f(x)$ at the endpoints. Thus, if the matrix

$$\mathcal{S}(f; [x_1, x_2]) = \begin{bmatrix} \text{sgn } f(x_1) \\ \text{sgn } f(x_2) \end{bmatrix}, \quad (2.16)$$

agrees with \mathcal{M}_1 , up to a permutation of the rows, then we say that $[x_1, x_2]$ is a *characteristic polyhedron*. Suppose now that $\Pi^n = \langle Y_1, Y_2, \dots, Y_{2^n} \rangle$ is an oriented n -dimensional polyhedron with 2^n vertices, $Y_k \in \mathbb{R}^n$ (i.e., an orientation has been assigned to its vertices), and let $F_n = (f_1, f_2, \dots, f_n): \Pi^n \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a nonlinear mapping from Π^n into \mathbb{R}^n . Then we call a *matrix of signs associated with F_n and Π^n* and denote it by $\mathcal{S}(F_n; \Pi^n)$, the $2^n \times n$ matrix whose entries in the k th row are the corresponding coordinates of the vector

$$\text{sgn } F_n(Y_k) = (\text{sgn } f_1(Y_k), \text{sgn } f_2(Y_k), \dots, \text{sgn } f_n(Y_k))^T. \quad (2.17)$$

That is,

$$\mathcal{S}(F_n; \Pi^n) = \begin{bmatrix} \text{sgn } F_n(Y_1) \\ \text{sgn } F_n(Y_2) \\ \vdots \\ \text{sgn } F_n(Y_k) \\ \vdots \\ \text{sgn } F_n(Y_{2^n}) \end{bmatrix}$$

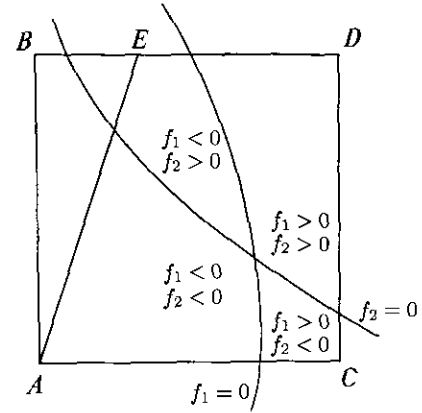


FIG. 3. $ABCD$ noncharacteristic polyhedron; $AECD$ characteristic polyhedron.

$$= \begin{bmatrix} \text{sgn } f_1(Y_1) & \text{sgn } f_2(Y_1) & \cdots & \text{sgn } f_n(Y_1) \\ \text{sgn } f_1(Y_2) & \text{sgn } f_2(Y_2) & \cdots & \text{sgn } f_n(Y_2) \\ \vdots & \vdots & \ddots & \vdots \\ \text{sgn } f_1(Y_k) & \text{sgn } f_2(Y_k) & \cdots & \text{sgn } f_n(Y_k) \\ \vdots & \vdots & \ddots & \vdots \\ \text{sgn } f_1(Y_{2^n}) & \text{sgn } f_2(Y_{2^n}) & \cdots & \text{sgn } f_n(Y_{2^n}) \end{bmatrix}. \quad (2.18)$$

The n -polyhedron $\Pi^n = \langle Y_1, Y_2, \dots, Y_{2^n} \rangle$ in \mathbb{R}^n is called a *characteristic n -polyhedron relative to $F_n = (f_1, f_2, \dots, f_n): \Pi^n \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$* , if the matrix of signs associated with F_n and Π^n , $\mathcal{S}(F_n; \Pi^n)$, is identical with the n -complete matrix \mathcal{M}_n . For example, the 2-polyhedron $AECD$ of Fig. 3 is a characteristic 2-polyhedron.

The i th *vertex* of Π^n is that vertex for which the coordinates of the vector $\text{sgn } F_n(Y_i)$ are identical to the corresponding entries of the i th row of \mathcal{M}_n . A *proper 1-simplex (edge)* of Π^n is an oriented 1-simplex $\langle Y_p, Y_q \rangle$ whose extreme points are vertices of Π^n for which the corresponding coordinates of the vectors, $\text{sgn } F_n(Y_p)$ and $\text{sgn } F_n(Y_q)$, differ from each other only in one entry. So, for each vertex Y_i of Π^n there are exactly n other vertices Y_k of Π^n such that the 1-simplexes $\langle Y_i, Y_k \rangle$ are proper 1-simplexes of Π^n , where the subindexes k are given by

$$k = i - 2^{n-j} C_{ij}, \quad j = 1, 2, \dots, n, \quad (2.19)$$

and where C_{ij} are the corresponding entries of \mathcal{M}_n (see [50] for a proof). It is easily seen that each such characteristic n -polyhedron has exactly $n2^{n-1}$ proper 1-simplexes.

Suppose now that Π^n is a characteristic n -polyhedron and that $F_n = (f_1, f_2, \dots, f_n): \Pi^n \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous. Then,

under suitable assumptions on the boundary of Π^n the value of the topological degree of F at Θ_n relative to Π^n is given by

$$\deg[F_n, \Pi^n, \Theta_n] = \pm 1 \neq 0 \quad (2.20)$$

(see [55, Theorem 2.9]).

The procedure for the construction of a characteristic n -polyhedron starts by picking an arbitrary point X^0 in \mathbb{R}^n ,

$$X^0 = (x_1^0, x_2^0, \dots, x_n^0)^\top, \quad (2.21)$$

with arbitrary stepsizes in each coordinate direction,

$$H = (h_1, h_2, \dots, h_n)^\top. \quad (2.22)$$

Using X^0 and H we now construct an initial n -polyhedron Π_0^n that is a scaled translation of the unit n -cube. Specifically, let \mathcal{G} be the rank-1, $n \times n$ matrix all of whose rows are equal to the row vector X^0 ; thus,

$$\mathcal{G} = [g_{ij}], \quad g_{ij} = x_j^0, \quad (2.23)$$

and let \mathcal{B} be the $n \times n$ diagonal matrix with elements h_1, h_2, \dots, h_n . Thus,

$$\mathcal{B} = [\beta_{ij}], \quad \beta_{ij} = \delta_i^j h_j, \quad (2.24)$$

where δ_i^j is the well-known Kronecker's delta. Also let the matrix

$$\begin{aligned} \mathcal{M}_n^* &= [C_{ij}^*], \quad C_{ij}^* = (C_{ij} + 1)/2, \\ i &= 1, 2, \dots, 2^n, \quad j = 1, 2, \dots, n, \end{aligned} \quad (2.25)$$

where C_{ij} are the elements of the n -complete matrix \mathcal{M}_n . Then the coordinates of the vertices of the initial n -polyhedron Π_0^n are given by the corresponding entries of the $2^n \times n$ matrix \mathcal{R} , where

$$\mathcal{R} = \mathcal{G} + \mathcal{M}_n^* \cdot \mathcal{B}. \quad (2.26)$$

For example, if $X^0 = (x_1^0, x_2^0)^\top$ and $H = (h_1, h_2)^\top$, the vertices of the rectangular $ABCD$ of Fig. 3 are

$$\begin{aligned} A &= (x_1^0, x_2^0)^\top, \quad B = (x_1^0, x_2^0 + h_2)^\top, \\ C &= (x_1^0 + h_1, x_2^0)^\top, \quad D = (x_1^0 + h_1, x_2^0 + h_2)^\top. \end{aligned} \quad (2.27)$$

The i th vertex of Π_0^n is the vertex of Π_0^n whose coordinates are identical to the corresponding entries of the i th row of the matrix \mathcal{R} . Of course, Π_0^n has exactly 2^n vertices. For example, the third vertex of the rectangle $ABCD$ of Fig. 3 is the vertex C . An edge of Π_0^n is an oriented 1-simplex whose extreme

points are vertices of Π_0^n , for example, $\langle V_p, V_q \rangle$, for which the corresponding coordinates of V_p and V_q differ from each other only in one coordinate. For example the edges of the rectangle $ABCD$ of Fig. 3 are the AB, AC, BD, CD . Of course, for each vertex of Π_0^n , for example, V_i , there are exactly n other vertices V_k of Π_0^n such that the 1-simplexes $\langle V_i, V_k \rangle$ are edges of Π_0^n . Moreover, on the basis of the relation that we have determined between the elements of \mathcal{M}_n and \mathcal{M}_n^* we easily observe that the subindexes k are also given by the relation (2.19) and that each Π_0^n has exactly $n2^{n-1}$ edges.

Now to construct a characteristic n -polyhedron we compare the matrix $\mathcal{S}(F_n; \Pi_0^n)$ with the matrix \mathcal{M}_n . If they are identical then Π_0^n is a characteristic n -polyhedron; otherwise the procedure creates "suitable" points X^s in \mathbb{R}^n (see below) such that their vectors of signs of F_n relative to these points produce the rows of \mathcal{M}_n that are missing in $\mathcal{S}(F_n; \Pi_0^n)$. For instance, the rectangle $ABCD$ of Fig. 3 is not a characteristic 2-polyhedron since the combination $(-1, 1)$ is missing, but if we replace the vertex B by the vertex E then the new polyhedron is a characteristic one. So, in general, we have to compute such points as the point E which we call them "suitable" points X^s . The points X^s lie in neighborhoods of the roots of the components of F_n lying on the edges of Π_0^n . More specifically, suppose that $\mathcal{A} = \{V_1, V_2, \dots, V_{2^n}\}$ is the ordered set of the vertices of Π_0^n and let $Z: \mathcal{A} \rightarrow \{1, 2, \dots, 2^n\}$ be the one-to-one function such that $Z(V_i) = i$ for all $i \in \{1, 2, \dots, 2^n\}$; then for the i th vertex of Π_0^n using (2.19) we can find n other subindexes k such that the 1-simplexes $\langle Z^{-1}(i), Z^{-1}(k) \rangle$ are edges of Π_0^n . Now for each one of the pairs (i, l) we assume the corresponding vertices,

$$\begin{aligned} V_i &= (v_1, v_2, \dots, v_m, \dots, v_n)^\top, \\ V_l &= (v_1, v_2, \dots, v'_m, \dots, v_n)^\top, \end{aligned} \quad (2.28)$$

of Π_0^n which by construction of Π_0^n are determined to have corresponding coordinates that differ from each other only in one entry, for example, in the m th. Next, by holding the $v_1, v_2, \dots, v_{m-1}, v_{m+1}, \dots, v_n$ fixed we solve the equations:

$$f_s(v_1, v_2, \dots, v_{m-1}, r_s, v_{m+1}, \dots, v_n) = 0, \quad s = 1, 2, \dots, n, \quad (2.29)$$

for r_s in the interval $(a, a + b)$ with $a = \min\{|v_m|, |v'_m|\}$ and $b = |v_m - v'_m|$ with an accuracy δ . Now suppose that for some s there is a solution r_s of Eqs. (2.29) in $(a, a + b)$; then the point

$$X^s = (v_1, v_2, \dots, v_{m-1}, r_s, v_{m+1}, \dots, v_n)^\top, \quad (2.30)$$

lies on the edge $\langle V_i, V_l \rangle$ of Π_0^n . Next we create the points

$$\begin{aligned} X_1^s &= (v_1, v_2, \dots, v_{m-1}, r_s + \delta^*, v_{m+1}, \dots, v_n)^\top, \\ X_2^s &= (v_1, v_2, \dots, v_{m-1}, r_s - \delta^*, v_{m+1}, \dots, v_n)^\top, \end{aligned} \quad (2.31)$$

where δ^* is a small positive number such that $\delta \leq \delta^* < \min\{r_s - a, (a + \beta - r_s)\}$. Finally if the $\text{sgn } F_n(X_u^s)$ for some $u = 1, 2$ coincides with any row of \mathcal{M}_n , for example, the λ th one, which was not present before in $\mathcal{S}(F_n; \Pi_0^n)$, the vertex V_λ of Π_0^n is replaced by the X_u^s . Now since F_n and Π_0^n are arbitrary, we do not know a priori for which edge of Π_0^n and for which component of F_n to do all this, until a characteristic n -polyhedron emerges. An order of the edges of Π_0^n which are going to be searched using the above process can be easily found (see [50, Lemma 4.2]).

We can use any one of the well-known one-dimensional methods to solve Eqs. (2.29). Here we use our modified scheme (2.8) of the traditional one-dimensional bisection method (see [40] for a discussion of its usefulness), since frequently the edges of Π_0^n are very long and only a few significant digits for the computation of the solutions of Eqs. (2.29) are required.

3. A BISECTION METHOD FOR COMPUTING PERIODIC ORBITS

In this section we briefly describe a generalized bisection method, used in combination with the criterion of Section 2, for the accurate computation of a periodic orbit of any period and accuracy. This method has all the advantages of the one-dimensional bisection method and is particularly useful in cases where the period of the orbit is very high, since it always converges within the initial specified region.

This method is based on the refinement of a characteristic n -polyhedron and is called a *characteristic bisection*. In the literature several bisection methods are available [18, 24, 28, 30, 41, 49] that require the computation of the topological degree in order to secure the nonzero value. In the bisection method described here, however, the computation of the topological degree is avoided by making sure that it retains a nonzero value at every iteration. The method bisects a characteristic n -polyhedron Π^n in such a way that the new refined n -polyhedron is also a characteristic one. To do this the method computes the midpoint of a proper 1-simplex (edge) of Π^n and uses it to replace that vertex of Π^n for which the vectors of their signs are identical. More specifically: Let $\langle X_i, X_j \rangle$ be a proper 1-simplex of Π^n and let $B = (X_i + X_j)/2$ be its midpoint. Then we distinguish the following three cases:

(1) If the vectors $\text{sign } F_n(B)$ and $\text{sign } F_n(X_i)$ are identical then we replace X_i by B and the process continues with the next proper 1-simplex.

(2) If the vectors $\text{sign } F_n(B)$ and $\text{sign } F_n(X_j)$ are identical then it replaces X_j by B and the process continues with the next proper 1-simplex.

(3) Otherwise the process continues with the next proper 1-simplex.

The above function, $\text{sign } \psi$, $\psi \in \mathbb{R}$, is

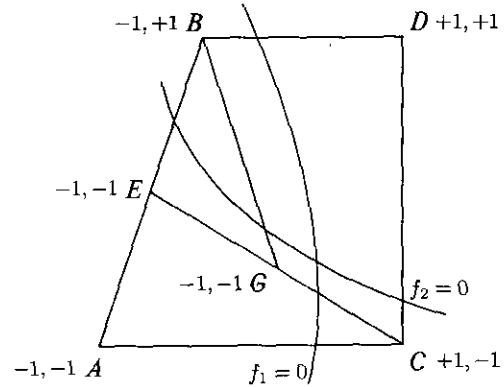


FIG. 4. Characteristic bisections applied to $ABCD$. The polyhedra $EBCD$, $GBCD$ are characteristic.

$$\text{sign } \psi = \begin{cases} -1, & \text{if } \psi < 0, \\ 1, & \text{if } \psi \geq 0. \end{cases} \quad (3.1)$$

For example, as we see in Fig. 4, one can subdivide the characteristic 2-polyhedron $ABCD$ in the following way: Starting from the edge AB we find its midpoint E and we calculate its vector of signs which in this case is $(-1, -1)$. So the vertex A is replaced by E in such a way that the new refined polyhedron $EBCD$ is also a characteristic one. Using the same procedure we are able to refine this polyhedron again by considering the midpoint G of EC and checking the vector of signs of this point. In this case, its vector of signs is $(-1, -1)$; hence the vertex E can be replaced by the vertex G , so that the new refined polyhedron $GBCD$ is also a characteristic one. This procedure continues until the midpoint of the longest diagonal of a refined polyhedron approximates the root within a predetermined accuracy.

In general, it is not difficult to show that, when a characteristic bisection is applied to a proper 1-simplex of Π^n , the new refined n -polyhedron is also a characteristic one (see [55] for a proof). Of course, an order of the proper 1-simplexes of Π^n that are going to be bisected using the characteristic bisection can be easily found using relation (2.19) as follows: Suppose that $\mathcal{A} = \{X_1, X_2, \dots, X_{2^n}\}$ is the ordered set of the vertices of Π^n and let $Z: \mathcal{A} \rightarrow \{1, 2, \dots, 2^n\}$ be the one-to-one function such that $Z(X_i) = i$ for all $i \in \{1, 2, \dots, 2^n\}$; then the 1-simplexes $\langle Z^{-1}(i), Z^{-1}(k) \rangle$, where

$$i = (t-1)2^{n-j} + m, \quad k = i + 2^{n-j}, \quad (3.2)$$

for all $j = 1, 2, \dots, n$, $t = 1, 3, \dots, 2^j - 1$, and $m = 1, 2, \dots, 2^{n-j}$ determine all the $n2^{n-1}$ proper 1-simplexes of Π^n (see [50, Lemma 4.2] for a proof). Now if, for the midpoint B of a proper 1-simplex of Π^n , $\langle X_i, X_j \rangle$, holds that $\text{sign } F_n(B) \neq \text{sign } F_n(X_i)$ and $\text{sign } F_n(B) \neq \text{sign } F_n(X_j)$ then we apply the *relaxation* or *reflection procedure* [48, 50, 55] which is briefly described

below. So, since $\text{sign } F_n(B)$ does not coincide with $\text{sign } F_n(X_i)$ and $\text{sign } F_n(X_j)$, it must coincide with $\text{sign } F_n(X_u)$ for some vertex X_u of Π^n such that $X_u \neq X_i$ and $X_u \neq X_j$. Now the relaxation process creates the point $X'_u = 2B - X_u$ and applies the characteristic bisection using X'_u instead of B . This process is applied repeatedly until for the point X'_u one of the following is true:

- (i) $\text{sign } F_n(X'_u) = \text{sign } F_n(X_i)$,
- (ii) $\text{sign } F_n(X'_u) = \text{sign } F_n(X_j)$;

otherwise it terminates when the number of its iterations becomes two.

Now suppose that $\langle X_1, X_2, \dots, X_{2^n} \rangle$ is the ordered set of vertices of a characteristic n -polyhedron Π^n ; then a *diagonal* of Π^n , say $\langle X_k, X_l \rangle$, is a 1-simplex such that all the corresponding components of the vectors $\text{sgn } F_n(X_k)$ and $\text{sgn } F_n(X_l)$ are different from each other. Now for each vertex of Π^n , X_k , there is exactly one other vertex X_l such that the 1-simplex $\langle X_k, X_l \rangle$ is a diagonal of Π^n , where the subindex l is given by $l = 2^n + 1 - k$ (see [50] for a proof). Of course each characteristic n -polyhedron has exactly 2^{n-1} diagonals. Moreover, we define the *diameter* of Π^n as the length of the longest proper 1-simplex of Π^n (where the distances are measured in Euclidean norms) and we denote it by $\Delta(\Pi^n)$, while the length of the longest 1-simplex of Π^n is called the *mesh* of Π^n and is denoted by $m(\Pi^n)$. Finally we define the midpoint of the longest diagonal of Π^n as an *estimate of a periodic orbit* and we denote it by P^* .

The number of characteristic bisections of the proper 1-simplexes of a Π^n required to obtain a new refined characteristic n -polyhedron Π_n^* such that $\Delta(\Pi_n^*) \leq \varepsilon$ for some $\varepsilon \in (0, 1)$ is given by

$$\xi = \lceil \log_2(\Delta(\Pi^n) \cdot \varepsilon^{-1}) \rceil \quad (3.3)$$

(see [55] for a proof). Based on the above formula we conjecture that our generalized bisection method is ‘‘optimal’’ and we hope to address the complete proof of this in a future publication. Furthermore, if $\Delta(\Pi^n) \leq \varepsilon$ holds for some Π^n , then the relations

$$\begin{aligned} m(\Pi^n) &\leq n\varepsilon, \\ \|P^* - P\|_2 &\leq n\varepsilon/2 \end{aligned} \quad (3.4)$$

are true (see [55] for a proof and [51] for extensions), where P in Π^n is such that $F_n(P) = \Theta_n$.

The above generalized method of bisection can become more efficient when the characteristic bisection is repeatedly applied to the diagonals starting with $\langle X_k, X_l \rangle$, for example, until $\text{sign } F_n((X_k + X_l)/2)$ becomes different than $\text{sign } F_n(X_k)$ and $\text{sign } F_n(X_l)$. One such order of diagonals is easily derived as follows: Suppose that $\mathcal{A} = \{X_1, X_2, \dots, X_{2^n}\}$ is the ordered set of the vertices of a characteristic n -polyhedron Π^n and let $Z: \mathcal{A} \rightarrow \{1, 2, \dots, 2^n\}$ be the one-to-one function such that $Z(X_i) = i$ for all $i \in \{1, 2, \dots, 2^n\}$; then the 1-simplexes $\langle Z^{-1}(k), Z^{-1}(l) \rangle$, where

$$l = 2^n + 1 - k \quad \text{for all } k = 1, 2, \dots, 2^{n-1}, \quad (3.5)$$

determine all the 2^{n-1} diagonals of Π^n (see [50, Lemma 4.3] for a proof).

The method described above does not require any computation of the topological degree; it requires only that the algebraic signs of the functions evaluations be known, so that it can be applied to problems with imprecise function values as well as to problems with function values that are large in magnitude. Thus, it is not affected by the mapping evaluations taking large values in neighborhoods of unstable periodic orbits. It is a globally convergent method. Also, it can be applied to nondifferentiable continuous functions F_n and does not involve derivatives of F_n or approximations of such derivatives.

4. LOCATING AND COMPUTING PERIODIC ORBITS

We now give a method for locating and computing periodic orbits of iterates of a nonlinear mapping to any predetermined accuracy. The method is illustrated here on the following quadratic area-preserving two-dimensional mapping,

$$\Phi_2: \begin{cases} \hat{x}_1 = x_1 \cos a - (x_2 - x_1^2) \sin a, \\ \hat{x}_2 = x_1 \sin a + (x_2 - x_1^2) \cos a. \end{cases} \quad (4.1)$$

To find all the fixed points of period p of Φ_2^p one usually solves a system of $2p$ equations for $2p$ unknowns [27, p. 295],

$$\left. \begin{aligned} x_{1,i+1} &= x_{1,i} \cos a - (x_{2,i} - x_{1,i}^2) \sin a \\ x_{2,i+1} &= x_{1,i} \sin a + (x_{2,i} - x_{1,i}^2) \cos a \end{aligned} \right\} \quad (i = 1, \dots, p), \quad (4.2)$$

with $x_{1,p+1} = x_{1,1}$, $x_{2,p+1} = x_{2,1}$. Now using the relation [27]

$$x_{2,i} = (x_{1,i-1} - x_{1,i} \cos a) / \sin a, \quad (4.3)$$

system (4.2) can be reduced to the system of p equations in p unknowns,

$$x_{1,i}^2 \sin a + 2x_{1,i} \cos a - x_{1,i-1} - x_{1,i+1} = 0, \quad i = 1, \dots, p, \quad (4.4)$$

with $x_{1,p} = x_{1,0}$, $x_{1,p+1} = x_{1,1}$. The solution of (4.4) gives all the fixed points of Φ_2^p .

Instead of applying the above procedure, we compute one periodic orbit of period p and then iterate the mapping in a proper way. More specifically, since an orbit X^* has period p this means that $\Phi_2^p(X^*) = X^*$, so we consider the mapping

$$F_2 = (f_1, f_2) = \Phi_2^p - I_2, \quad (4.5)$$

where I_2 is the identity mapping, and we solve for any period p the following system of two equations in two unknowns,

$$F_2(X) = \Theta_2 = (0, 0)^T. \quad (4.6)$$

To do this we choose a starting point,

$$X^0 = (x_1^0, x_2^0)^T, \quad (4.7)$$

and two stepsizes in each coordinate direction,

$$H = (h_1, h_2)^T, \quad (4.8)$$

in such a way that the corresponding constructed box (see Fig. 3) forms a domain in which the method will attempt to locate and compute a solution of the system $F_2(X) = \Theta_2$, which is periodic orbit of the mapping Φ_2^p .

Suppose now that a periodic point X_i^* of the orbit has been computed within a predetermined accuracy ε such that

$$\|\Phi_2^p(X_i^*) - X_i^*\| \leq \varepsilon. \quad (4.9)$$

Then, in order to compute all the other points X_i^* , $i = 2, \dots, p$, of the orbit with the same period and accuracy ε we iterate the mapping Φ_2 as follows: First we obtain an approximation \hat{X}_2 of the next point X_2^* of the orbit by the relation

$$\hat{X}_2 = \Phi_2(X_i^*) \quad (4.10)$$

and check if the following relation is fulfilled:

$$\|\Phi_2^p(\hat{X}_2) - \hat{X}_2\| \leq \varepsilon. \quad (4.11)$$

If so, then we choose $X_2^* = \Phi_2(X_i^*)$, and the procedure continues with the verification of the next periodic point X_3^* taking as an initial estimate the value of $\Phi_2(X_2^*)$; otherwise, the procedure constructs a box which contains the 2-ball centered at \hat{X}_2 with a radius $\eta\|\Phi_2^p(\hat{X}_2) - \hat{X}_2\|$ for some $\eta \in \mathbb{R}$ and applies the generalized bisection method of the previous section to it, in order to obtain the point X_2^* to the accuracy ε . When such a point is verified the procedure continues with the next point repeatedly until the last point X_p^* is obtained.

Of course, we can apply the bisection method for computing a periodic point X_i^* of the orbit within the above small box by choosing a starting point X^0 close to \hat{X}_i and small stepsizes H in such a way as to include the X_i^* in this box. For instance, we can apply the bisection method at the outset for the improvement of the accuracy of an approximate periodic orbit \hat{X}_i , using the starting point

$$X^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} \hat{x}_{1,i} - \eta_1\|\Phi_2^p(\hat{X}_i) - \hat{X}_i\| \\ \hat{x}_{2,i} - \eta_2\|\Phi_2^p(\hat{X}_i) - \hat{X}_i\| \end{pmatrix}, \quad (4.12)$$

(where $\hat{x}_{1,i}$ and $\hat{x}_{2,i}$ determine the corresponding coordinates of \hat{X}_i) and taking the stepsizes

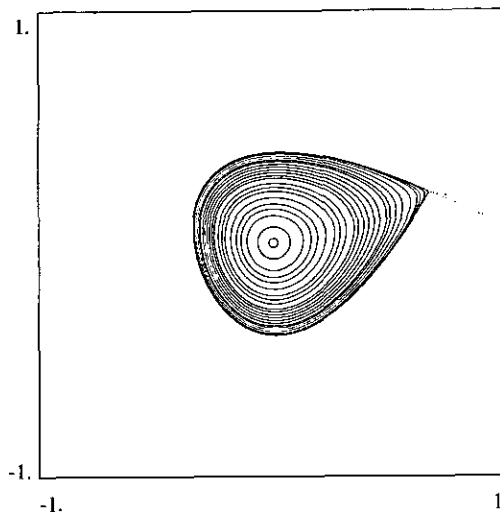


FIG. 5. Hénon's mapping for $\cos a = 0.8$. This figure as well as Figs. 6–10 are drawn using the new software package GIOTTO of [56].

$$H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 2\eta_1\|\Phi_2^p(\hat{X}_i) - \hat{X}_i\| \\ 2\eta_2\|\Phi_2^p(\hat{X}_i) - \hat{X}_i\| \end{pmatrix}, \quad (4.13)$$

where $\eta_1, \eta_2 \in \mathbb{R}$ are constants. Our experience is that we can obtain in this way all the stable periodic orbits within a predetermined accuracy of ε by iterating the mapping using the bisection method only once, even for high periods ($p > 1000$). In the case of the unstable periodic orbits, however, this is in general true for low periods ($p < 1000$), while for high periods we have to apply more than once the bisection method to the above described tiny box in order to obtain all the periodic points within the desired accuracy.

Let us illustrate the above procedure on Hénon's mapping (4.1) for different values of a and various periods. In general, a visualization of the orbits of the mapping is very helpful for choosing the starting point X^0 and the stepsizes H (see [56] for such a visualization). In any case if such a visualization is not available, one can search within various boxes or take a suitable grid for the domain of interest. The phase plots (Figs. 5–10) which are shown here are drawn using the new software package GIOTTO of [56].

First we applied our method to orbits, which are, in general, difficult to compute otherwise. For example, for $a = \cos^{-1}(0.8)$ the simple hyperbolic fixed point of Figs. 5 and 6 is very hard to find numerically. On the other hand, such points are very useful, since one can follow the invariant manifolds of this hyperbolic fixed point to estimate the dynamic aperture (or stability domain) of such nonlinear systems as one finds in high energy accelerators [4, 19].

Now in order to find the hyperbolic fixed point in this case we include it into a box by taking the appropriate values for X^0 and H . Thus if we take

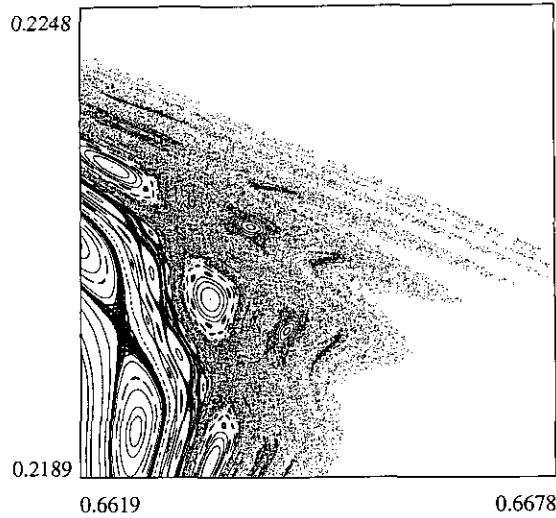


FIG. 6. Magnification of a region of the single hyperbolic fixed point of Fig. 5.

$$X^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.2 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix},$$

and accuracy $\varepsilon = 10^{-16}$, we easily compute

$$X_1^1 = (0.6666666666666666, 0.2222222222222222)^T,$$

utilizing around 350 μ s of CPU time on the CERN VAX 9000-410 system. Of course, one can easily verify that this point is indeed unstable by applying the traditional technique of checking the eigenvalues of the linearized mapping [26, p. 353].

We would like to discuss here the reason why our method is efficient in this kind of problem. Looking at the phase plot of Fig. 6 we can observe that the mapping iterates are repelled to infinity. On the other hand, as we said previously, the only computable information required by our method is the algebraic signs of the components of the mapping, which, of course, are not affected by the mapping evaluations taking large values. Thus our method will converge within the starting box by bisecting it properly, independently of the magnitude of these function evaluations.

Let us now apply this procedure to compute periodic orbits of higher periods. Taking, for example, $a = \cos^{-1}(0.24)$ we can see in the phase plot of Hénon's mapping (see Fig. 7) that there is a chain of five big islands around the origin. So in this case we shall search for five elliptic and five hyperbolic periodic points of period $p = 5$. The reason for this choice of a is that the corresponding phase plot has a large region of stability around the origin (see [8, p. 1868; 26, p. 364; 27, p. 298]) and it may be of interest to applications in beam dynamics.

Now in order to find the elliptic periodic orbit of this period we choose one island, include it into a box by taking the appropriate values for X^0 and H , e.g.,

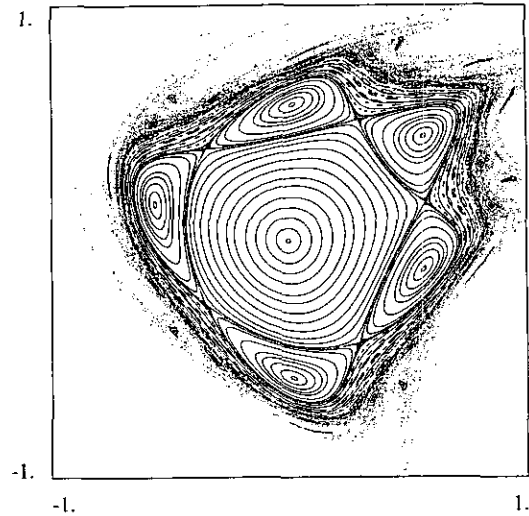


FIG. 7. Hénon's mapping for $\cos a = 0.24$.

$$X^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 0.3 \\ -0.3 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix},$$

and apply our method with accuracy $\varepsilon = 10^{-16}$. In this case, we compute the following stable fixed point utilizing around 6.5 ms of CPU time on the CERN VAX 9000-410 system:

$$X_1^1 = (0.5672405470221847, -0.1223202134278941)^T.$$

Now we can proceed with the computation of the rest of the stable points of period five. To do this, we either apply the bisection method or iterate the mapping, since we have computed one of them already within the predetermined accuracy $\varepsilon = 10^{-16}$. We prefer to do the latter and we successively compute the points

$$X_2^1 = (0.5672405470221847, 0.4440820516139216)^T,$$

$$X_3^1 = (0.0173925844399303, 0.5800185952239573)^T,$$

$$X_4^1 = (-0.5585984457571741, 0.1560161118011652)^T,$$

$$X_5^1 = (0.0173925844399305, -0.5797160932304572)^T,$$

utilizing around 70 μ s of CPU time on the same machine.

It is well known that periodic orbits are identified by their rotation number σ ,

$$\sigma = \frac{\nu}{2\pi} = \frac{m_1}{m_2}, \quad (4.14)$$

where ν is the frequency of the orbit and m_1, m_2 , are two positive integers [9, 23]. So from the sequence with which the above points are created on the x_1, x_2 plane, we can infer the

rotation number of this orbit $\sigma = m_1/m_2 = 1/5$, indicating that it has produced $m_2 = 5$ points, by rotating around the origin $m_1 = 1$ times.

Now to compute the unstable periodic orbit of period $p = 5$ we choose the initial values

$$X^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 0.1 \\ -0.7 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix},$$

and apply our method with the same accuracy, $\varepsilon = 10^{-16}$, to find one of its points. Thus, the following point is computed utilizing around 6.5 ms,

$$X_1^5 = (0.2942106885737921, -0.4274862418615337)^T.$$

Then, by iterating the mapping, we compute the rest points of the unstable periodic orbit of period $p = 5$, within the predetermined accuracy $\varepsilon = 10^{-16}$,

$$X_2^5 = (0.5696326513533602, 0.1622406787439296)^T,$$

$$X_3^5 = (0.2942106885737916, 0.5140461711325987)^T,$$

$$X_4^5 = (-0.3443814883177751, 0.3882084578625210)^T,$$

$$X_5^5 = (-0.3443814883177746, -0.2696098483665559)^T.$$

utilizing around 70 μ s of CPU time on the same machine.

Looking at the phase plot (Fig. 7) of Hénon's mapping for the same value of a as before, we are able to distinguish 16 islands across the "boundary" of the mapping. Hence, in this case, we search for 16 stable and 16 unstable fixed points of period $p = 16$. To do this we choose one island and enclose it into a box by choosing appropriate values for X^0 and H . Thus, by taking

$$X^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.1 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix},$$

and applying our method (with accuracy of $\varepsilon = 10^{-16}$), we compute the following stable periodic orbit utilizing around 11.1 ms,

$$X_1^{16} = (0.8504309709743801, 0.1490801034942473)^T.$$

We may now proceed with the computation of the rest of the points of the stable periodic orbit of period 16 within the same accuracy. Iterating the mapping we successively find the remaining stable points of the periodic orbit within the predetermined accuracy of $\varepsilon = 10^{-16}$, utilizing around 0.5 ms,

$$X_2^{16} = (0.7614753396800761, 0.6877786738086518)^T,$$

$$X_3^{16} = (0.0779746992070990, 0.7651237700205194)^T,$$

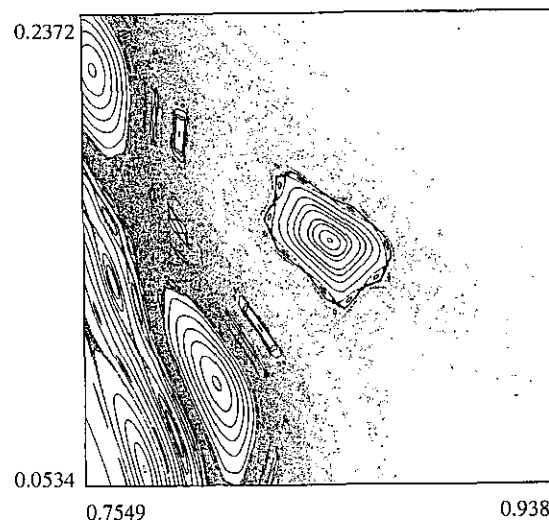


FIG. 8. Magnification of an island of period 16 of Hénon's mapping for $\cos a = 0.24$.

$$X_4^{16} = (-0.7181451327554049, 0.2578662158501398)^T,$$

$$X_5^{16} = (0.0779746992070976, -0.7590437163040815)^T,$$

$$X_6^{16} = (0.7614753396800752, -0.1079339808677657)^T,$$

$$X_7^{16} = (0.8504309709743798, 0.5741527328981787)^T,$$

$$X_8^{16} = (0.3488263556404760, 0.7897961048655951)^T,$$

$$X_9^{16} = (-0.5648708437963138, 0.4989790754962736)^T,$$

$$X_{10}^{16} = (-0.3102110502247625, -0.5051852990790463)^T,$$

$$X_{11}^{16} = (0.5093878842002730, -0.4454843638466283)^T,$$

$$X_{12}^{16} = (0.8066095166031799, 0.3253094561374075)^T,$$

$$X_{13}^{16} = (0.5093878842002735, 0.7049603804166589)^T,$$

$$X_{14}^{16} = (-0.3102110502247618, 0.6014161947605968)^T,$$

$$X_{15}^{16} = (-0.5648708437963144, -0.1799000053251131)^T,$$

$$X_{16}^{16} = (0.3488263556404757, -0.6681162784761799)^T.$$

Note that, from the sequence with which these points are created on the x_1, x_2 plane, we can infer the rotation number of this orbit,

$$\sigma = m_1/m_2 = 3/16,$$

indicating that it has produced $m_2 = 16$ points by rotating around the origin $m_1 = 3$ times.

We have also applied the above procedure to compute periodic orbits of higher periods. Enlarging the vicinity of an island of period 16, we distinguish in Fig. 8 a chain of nine islands around it. So the period of each one of these islands is $p =$

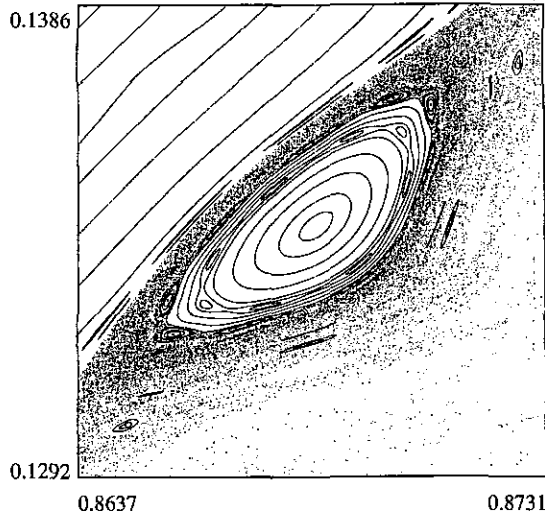


FIG. 9. Magnification of an island of period 144 of Hénon's mapping for $\cos a = 0.24$.

$16 \times 9 = 144$. To compute all the points of this period, let us start with the computation of one which is stable. Taking

$$X^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 0.865 \\ 0.131 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0.005 \\ 0.005 \end{pmatrix}$$

and applying our method within accuracy $\varepsilon = 10^{-16}$, we computed the point utilizing 46 ms,

$$X_1^{144} = (0.8685006897387088, 0.1341772208865609)^T.$$

The other points of this orbit are obtained as before by iterating the mapping and achieving an accuracy of $\varepsilon = 10^{-16}$ utilizing around 6 ms of CPU time.

By the same procedure we have also computed the corresponding unstable orbit of period 144, forming a "chain" of

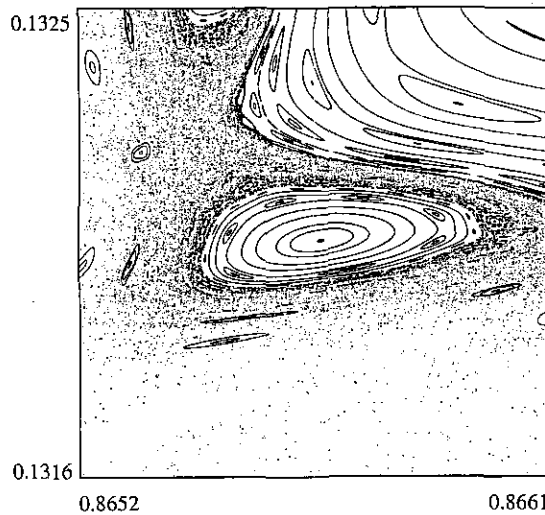


FIG. 10. Magnification of an island of period 1296 of Hénon's mapping for $\cos a = 0.24$.

TABLE I

Periodic Orbits of Hénon's 2D Mapping and Total CPU Time for the Computation of All Their Periodic Points

Period p	x_1	x_2	CPU time
1	0.6666666666666666	0.2222222222222222	350 μ s
5	0.5672405470221847	0.4440820516139216	6.5 ms
16	0.8504309709743801	0.1490801034942473	11.6 ms
144	0.8685006897387088	0.1341772208865609	52 ms
1296	0.8656539452140570	0.1320418594308142	2.25 s
10368	0.8655111610927791	0.1321457880746314	5.9 s

unstable periodic orbits located in between the stable periodic orbits and having the same rotation number $\sigma = \frac{27}{144}$ as the stable orbit. Thus, choosing

$$X^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 0.863 \\ 0.126 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0.005 \\ 0.005 \end{pmatrix},$$

and accuracy $\varepsilon = 10^{-16}$, we find the point utilizing almost 46 ms,

$$X_1^{144} = (0.8631963046006437, 0.1304179001540082)^T.$$

The other points of this orbit are obtained as before by iterating the mapping and achieving an accuracy of $\varepsilon = 10^{-16}$ utilizing around 6 ms of CPU time.

Enlarging the vicinity of an island of period 144 further, we distinguish in Fig. 9 a chain of nine islands around it whose period is $p = 144 \times 9 = 1296$. Finally, by enlarging the vicinity of an island of this period we distinguish in Fig. 10 a chain of eight islands around it, with period $p = 1296 \times 8 = 10368$, where each one has a chain of six islands around it, with period $p = 10368 \times 6 = 62208$. The rotation numbers of these orbits, respectively, are $\sigma = \frac{243}{1296}, \frac{1944}{10368}, \frac{11664}{62208}$ and the coordinates of some of their points are listed in Table I. The calculation of all the corresponding periodic points of unstable orbits requires total CPU times of the same, if not smaller, magnitude.

It is now clear, from the above, that one can similarly proceed to calculate orbits of higher and higher period in CPU times which do not grow so rapidly as to make the calculation of these orbits impractical (see Table I).

5. COMPUTING PERIODIC ORBITS OF A FOUR-DIMENSIONAL MAPPING

In this section we apply our method to the computation of periodic orbits of period p of Hénon's mapping (4.1) with complex components. We say that a point $A + Bi$, ($i = \sqrt{-1}$), where $A, B \in \mathbb{R}^2$, belongs to a *complex periodic orbit* of a mapping Φ_2^p of period p if

$$\Phi_2^p(A + Bi) = A + Bi. \tag{5.1}$$

In order to compute such complex periodic orbits we expand a given two-dimensional mapping to four dimensions and form a real four-dimensional mapping, separating real and imaginary parts. Then, for a particular period p orbit we consider the mapping

$$F_4 = (f_1, f_2, f_3, f_4) = \Phi_4^p - I_4, \quad (5.2)$$

where I_4 is the identity mapping and solve the following system of four equations in four unknowns,

$$F_4(X) = \Theta_4 = (0, 0, 0, 0)^T. \quad (5.3)$$

In the case of Hénon's mapping (4.1), replacing x_1, x_2 by their complex form, $x_j + iy_j, j = 1, 2, x_j, y_j \in \mathbb{R}$, yields the following system of equations,

$$\begin{aligned} \hat{x}_1 &= x_1 \cos a - (x_2 - x_1^2 + y_1^2) \sin a, \\ \hat{y}_1 &= y_1 \cos a - (y_2 - 2x_1y_1) \sin a, \\ \hat{x}_2 &= x_1 \sin a + (x_2 - x_1^2 + y_1^2) \cos a, \\ \hat{y}_2 &= y_1 \sin a + (y_2 - 2x_1y_1) \cos a, \end{aligned} \quad (5.4)$$

which can be written as

$$\begin{aligned} \Phi_4: \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} &= \begin{pmatrix} \cos a & -\sin a & 0 & 0 \\ \sin a & \cos a & 0 & 0 \\ 0 & 0 & \cos a & -\sin a \\ 0 & 0 & \sin a & \cos a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 - x_1^2 + y_1^2 \\ y_1 \\ y_2 - 2x_1y_1 \end{pmatrix}, \end{aligned} \quad (5.5)$$

for a given value of the frequency a .

In this case, of course, a visualization of these four-dimensional orbits is quite a complicated problem. For our purposes, it is convenient to consider two-dimensional projections of these orbits and use them to obtain good initial values of X^0 and H . If that is not easily achieved, we may also take a grid in which the methods can be applied for the computation of the periodic orbits. Obviously, we can also iterate here the mapping (5.5) for one point of a computed periodic orbit to find all the other points of the orbit, for a particular period p .

Let us apply our method to mapping (5.5) to compute periodic orbits for a given value of a and period p . We would like to mention here that the following results are obtained on a PC IBM compatible (with an i286 processor) so that the computational times are not compatible with the previous results. In any case, the results are still quite rapidly obtained. First, we

choose $a = \cos^{-1}(0.24)$ and $p = 2$ and select a proper grid to calculate within the accuracy of $\varepsilon = 10^{-16}$ the following complex periodic orbit of period 2:

$$X_1^2 = \begin{pmatrix} -1.2773327473170111 \\ -1.0000000000000000 \\ 1.9056702094980712 \\ -2.4341749641783542 \end{pmatrix}.$$

Now we proceed to the computation of the second point of the orbit, by iterating the mapping (5.5) to obtain

$$X_2^2 = \begin{pmatrix} -1.2773327473170112 \\ -0.9999999999999999 \\ -1.9056702094980711 \\ 2.4341749641783541 \end{pmatrix}.$$

Iterating the mapping (5.5) for X_2^2 produces the point X_1^2 , which verifies that this periodic orbit has indeed period 2.

Proceeding in the same way, we have been able to compute, using our method, periodic orbits of various periods. For example, choosing $a = \cos^{-1}(0.24)$ and $p = 3$ and taking a proper grid we have computed within the accuracy of $\varepsilon = 10^{-16}$ the following orbit of period 3:

$$X_1^3 = \begin{pmatrix} -1.2773327473170111 \\ 0.0611205432937182 \\ 1.2285511493682861 \\ -1.5692685842514041 \end{pmatrix}.$$

Once this point is known, one can proceed to the computation of the other two points by iterating the mapping:

$$X_2^3 = \begin{pmatrix} -0.2472257569381386 \\ -1.2546689145905351 \\ -1.2285511790648683 \\ 1.5692686221838227 \end{pmatrix},$$

$$X_3^3 = \begin{pmatrix} -0.2472257569381386 \\ -0.1935484196177614 \\ -1.2285511790648683 \\ -0.9618098644716342 \end{pmatrix}.$$

Similarly, for $a = \cos^{-1}(0.24)$ and accuracy $\varepsilon = 10^{-16}$ we have obtained higher period orbits as, for example, the following which have periods $p = 4, 33, 37, 137, 1297$, respectively:

$$\begin{aligned}
X_1^4 &= \begin{pmatrix} -0.2472256930290987 \\ -0.1935483870967742 \\ 0.6694903415937131 \\ -0.8551619373301012 \end{pmatrix}, \\
X_1^{33} &= \begin{pmatrix} -0.0523586445910415 \\ -0.0752756616334571 \\ -0.1378214025236530 \\ 0.1061106148479258 \end{pmatrix}, \\
X_1^{37} &= \begin{pmatrix} -0.0626633697827437 \\ -0.1041456979321811 \\ -0.3287913635358173 \\ 0.2450616114372278 \end{pmatrix}, \\
X_1^{37} &= \begin{pmatrix} 0.1120259657454833 \\ -0.1445554751408692 \\ 0.1465891753951452 \\ 0.1504948991894342 \end{pmatrix}, \\
X_1^{297} &= \begin{pmatrix} -0.1634028690569280 \\ 0.0346963745030925 \\ -0.0023777665715609 \\ -0.0165731199145865 \end{pmatrix}.
\end{aligned}$$

The remaining points of these periodic orbits can be obtained as before by iterating the mapping.

Note that the mapping (5.5), while conservative (the determinant of its Jacobian is unity [8, 17, 23]) it is not symplectic [3, 31, 33]. Our method, however, can also be easily applied to periodic orbits of $2N$ -dimensional symplectic mappings, with $N > 1$. In fact, we have already carried out such a generalization and computed some of the periodic orbits of a four-dimensional symplectic mapping, which has two frequencies, a_1 and a_2 , and is closely related to (5.5):

$$\begin{aligned}
\Phi_4: \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{pmatrix} &= \begin{pmatrix} \cos a_1 & -\sin a_1 & 0 & 0 \\ \sin a_1 & \cos a_1 & 0 & 0 \\ 0 & 0 & \cos a_2 & -\sin a_2 \\ 0 & 0 & \sin a_2 & \cos a_2 \end{pmatrix} \\
&\cdot \begin{pmatrix} x_1 \\ x_2 + x_1^2 - x_3^2 \\ x_3 \\ x_4 - 2x_1x_3 \end{pmatrix}. \tag{5.6}
\end{aligned}$$

This is an interesting model, since mappings of this form are of direct relevance to the dynamics of particle beams (in a four-dimensional phase space) passing repeatedly through FODO cells of magnetic focusing elements [9]. In this context the computation of high period orbits can be quite useful in determining the existence and structure of nearby invariant surfaces, which are composed of bounded, quasiperiodic orbits. More detailed results in this direction, however, are expected to appear in a future publication [57].

6. CONCLUDING REMARKS AND DISCUSSION

In this paper, we have described an efficient method for rapidly and accurately locating and computing periodic orbits (stable, unstable, and complex) of nonlinear mappings of dynamical systems to any desired accuracy and period. This method exploits topological degree theory to provide a criterion, by constructing a characteristic polyhedron for the localization of a periodic orbit of an iterate of the mapping within a given region, without making any computation of the topological degree. Then it subdivides this polyhedron, by avoiding calculations concerning the topological degree, to compute the isolating periodic orbit rapidly and to any accuracy.

The method is very efficient, since the only computable information that is required is the algebraic signs of the components of the mapping. Thus it is not affected by the mapping evaluations taking large or imprecise values. Moreover, it always converges rapidly to a periodic orbit within the initial specified region independently of the initial guess, which is particularly useful in cases where the period of the periodic orbit is very high and the mapping has many periodic orbits, stable and unstable, close to each other.

It is also a globally convergent method, as well as that it can be applied to nondifferentiable continuous functions and does not involve derivatives or approximations of such derivatives. Furthermore, using this method it is easy to find out the number of iterations needed for the attainment of an approximate periodic orbit to a predetermined accuracy. Based on this we conjecture that the method of the present paper is optimal (i.e., that it possesses asymptotically the best rate of convergence) and we hope to address the complete proof of this in a future publication.

We have illustrated this method first to Hénon's two-dimensional mapping used in the study of beam dynamics in particle accelerators and have succeeded in overcoming the difficulties of other schemes in the computation of periodic orbits which are strongly unstable and/or of very high period.

Moreover, we have applied our method to higher dimensions, by using it to calculate periodic orbits of four-dimensional nonlinear mappings of dynamical systems with physical applications.

Since the method is especially suited for the calculation of orbits of high period, it can be used to approximate quasiperiodic orbits which lie on invariant tori of nonlinear mappings.

Thus, we may be able to check in the case of four (and higher) dimensions the validity of the famous Greene's conjecture concerning the breakup of invariant surfaces and the onset of large scale chaos [23]. These topics will be addressed in more detail in a future publication.

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